Quantum critical transport in clean graphene

Lars Fritz,[1] Jörg Schmalian,[2] Markus Müller,[1] and Subir Sachdev[1]

[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*Ames Laboratory and Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011, USA*

We describe electrical transport in ideal single-layer graphene at zero applied gate voltage. There is a crossover from collisionless transport at frequencies larger than $k_B T / \hbar$ (T is the temperature) to collision-dominated transport at lower frequencies. The dc conductivity is computed by the solution of a quantum Boltzmann equation. Due to a logarithmic singularity in the collinear scattering amplitude (a consequence of relativistic dispersion in two dimensions), quasiparticles and quasiholes moving in the same direction tend to an effective equilibrium distribution whose parameters depend on the direction of motion. This property allows us to find the nonequilibrium distribution functions and the quantum critical conductivity exactly to leading order in $1/|\ln(\alpha)|$, where α is the coupling constant characterizing the Coulomb interactions.

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I. INTRODUCTION

Despite the intense experimental and theoretical interest in the electronic properties of graphene,[1] there has been relatively little progress in measuring and understanding the role of electron-electron interactions. However, the recent ability to grow ultrahigh mobility, suspended, single-layer graphene[2,3] promises that the situation may well change in the near future.

This paper will examine the role of electron-electron interactions in an infinite sample of single-layer graphene without impurities. We will also restrict our attention to the undoped case, so that the chemical potential is at the node of the massless Dirac spectrum. Our results can be extended to include a nonzero chemical potential and a dilute concentration of impurities: this was discussed recently in Ref. 4 for a low-frequency "hydrodynamic" regime, and additional results will appear in forthcoming work.

The key to understanding electron-electron interactions in clean, undoped graphene is the fact that it is a nearly "quantum critical" system with marginally irrelevant Coulomb interactions.[5–9] This implies that the inelastic electron-electron-scattering rate is of order $k_B T / \hbar$, where T is the absolute temperature, and there is a crossover from hydrodynamic to "collisionless" transport as the measurement frequency (ω) is increased past the scattering rate.[10,11] These two regimes are captured in the following limiting forms for the frequency dependence of the electrical conductivity, σ:

$$\sigma(\omega)$$
$$= \begin{cases} \dfrac{e^2}{h} \left\{ \dfrac{\pi}{2} + \mathcal{O}\left[\dfrac{1}{\ln(\Lambda/\hbar\omega)} \right] \right\}, & \hbar\omega \gg k_B T \\[2ex] \dfrac{e^2}{h\alpha^2(T)} \left\{ 0.760 + \mathcal{O}\left[\dfrac{1}{|\ln(\alpha(T))|} \right] \right\}, & \hbar\omega \ll k_B T \alpha^2(T), \end{cases}$$

$$(1.1)$$

where $\alpha(T)$ is a temperature-dependent, dimensionless "fine-structure constant," which controls the strength of the electron-electron interactions (defined more precisely in Sec. II), and Λ is a cutoff energy scale of the order of the electronic bandwidth. The high-frequency result above (the collisionless regime) was obtained in Refs. 9, 12, and 13. The leading term is the conductivity of four species of free massless Dirac fermions. Herbut *et al.*[13] also obtained the coefficient of the subleading $[\ln(\Lambda/\hbar\omega)]^{-1}$ term. The low-frequency result in Eq. (1.1), which corresponds to the collision-dominated hydrodynamic regime, is the primary result of this paper. At asymptotically low temperatures we have [see Eq. (2.4)]

$$\alpha(T) \approx \frac{4}{\ln(\Lambda/T)}. \qquad (1.2)$$

The resulting logarithmic increase in σ with decreasing T is similar to those of quantum critical systems at their upper-critical dimension.[14] This can explicitly be seen in Fig. 1, where the dc conductivity is plotted as a function of T; Fig. 2 shows a plot of the ac conductivity as a function of $\hbar\omega/(k_B T)$. We see in Eq. (1.1) that the inelastic-scattering rate of the carriers is of order $\tau^{-1} \sim (k_B T / \hbar) \alpha^2(T)$. This is closely related to the finding of González *et al.*[6] that the inverse lifetime of quasiparticle excitations due to Coulomb
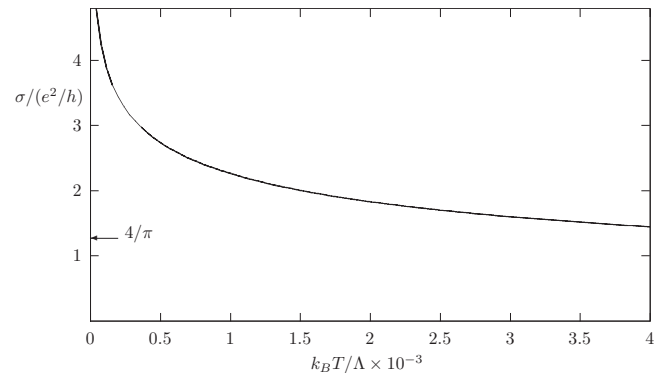

FIG. 1. We show the dc conductivity with $\alpha(T)$ from Eq. (1.2) as a function of T. The regime shown corresponds to $T=0$ K up to approximately room temperature for a reasonable cutoff Λ of several eV. The arrow indicates the result of the relaxation-time approximation to the Boltzmann equation in the limit $\hbar\tau^{-1} \gg k_B T$.
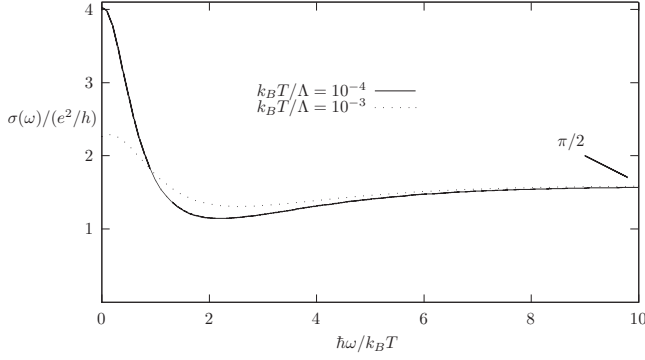
FIG. 2. We show the ac conductivity as a function of $\hbar\omega/k_B T$ for different T. The low-frequency regime ($\hbar\omega/k_B T < 1$) corresponds to the hydrodynamic regime studied in this paper. In distinction, the high-frequency limit $\sigma(\hbar\omega/k_B T \to \infty) = \frac{\pi}{2}\frac{e^2}{h}$ does not depend on the electron-electron interaction. The marginally irrelevant Coulomb interaction leads to violations of a perfect ω/T scaling.

interactions is proportional to their energy. Our result for conductivity (1.1) can then be qualitatively understood by inserting this lifetime into the expression for the conductivity given in an early paper by Wallace,[15] $\sigma = A e^2/h * (k_B T\tau/\hbar)$. Related results have been obtained recently by Kashuba[16] in a preprint which appeared while our paper was being completed.

We note that our results are obtained in the context of a solution of the quantum Boltzmann equation. Going beyond the Boltzmann approximation and in a system with perfect momentum conservation, it is expected[17] that hydrodynamic "long-time tails" will induce a weak logarithmic dependence upon $\hbar\omega/(k_B T)$ for $\hbar\omega \ll k_B T\alpha^2(T)$.

Before we proceed to the calculation, let us explain how a finite conductivity can come about in a pure system. Indeed, the latter appears surprising in the light of transport in conventional clean Fermi liquids, where translational invariance implies momentum conservation and entails the persistence of currents in spite of electron-electron scattering. In general one thus expects an infinite conductivity in pure systems. In a Dirac liquid, however, where there is an equal number of particles and holes, momentum conservation does not imply current conservation. Physically, electrons and holes are accelerated into opposite directions creating no net momentum, while the currents add up, going in the same direction. The current relaxation arises from the scattering of electrons from holes and vice versa, while the net momentum always vanishes.

II. RENORMALIZATION-GROUP ANALYSIS

Here, and in the remainder of the paper, we set $\hbar = k_B = 1$. We begin by introducing the low-energy theory for graphene and review its renormalization-group (RG) properties. The theory is expressed in terms of $N = 4$ species of two-component Dirac fermions Ψ_a ($a = 1, \ldots, N$) and the Euclidean partition function,

$$\mathcal{Z} = \int \mathcal{D}\Psi_\alpha \mathcal{D}A_\tau \exp(-\mathcal{S}),$$

$$\begin{aligned}
\mathcal{S} = \sum_{a=1}^{N} \int d\mathbf{x} \int d\tau \Psi_a^\dagger(\mathbf{x}, \tau) &\left[\frac{\partial}{\partial \tau} + ieA_\tau(\mathbf{x}, \tau) \right. \\
&+ iv_F^0\sigma^x\left(\frac{\partial}{\partial x} + i\frac{e}{c}A_x \right) + iv_F^0\sigma^y\left(\frac{\partial}{\partial y} + i\frac{e}{c}A_y \right) \left. \right] \Psi_a(\mathbf{x}, \tau) \\
&+ \frac{1}{2} \int \frac{d^2q}{4\pi^2} \int d\tau \frac{\varepsilon q}{2\pi} |A_\tau(\mathbf{q}, \tau)|^2.
\end{aligned}$$

The functional integral is over fields defined in two spatial dimensions $\mathbf{x} = (x, y)$ and imaginary time τ, $\sigma^{x,y}$ are Pauli matrices acting on the sublattice space of the honeycomb lattice, and v_F^0 is the bare Fermi velocity. The scalar potential, A_τ, mediates the $e^2/(\varepsilon|\mathbf{x}|)$ Coulomb interaction between the electrons, where $\varepsilon = (\varepsilon_A + \varepsilon_B)/2$ is the dielectric constant for a graphene sheet confined between two dielectrica with dielectric constants ε_A and ε_B. We have also introduced a nonfluctuating external vector potential $\mathbf{A} = (A_x, A_y)$ as a source field: this allows us to extract the electrical current.

The renormalization-group properties of $\mathcal{Z}$ have been discussed elsewhere.[5–9] The fermion field Ψ_a undergoes a wave-function renormalization, the charge e remains unrenormalized, and the velocity v_F renormalizes to larger values with decreasing energy scale. For the velocity renormalization, we have the RG equation,

$$\frac{dv_F}{d\ell} = f(\alpha)v_F, \tag{2.1}$$

where the running fine-structure constant is

$$\alpha \equiv \frac{e^2}{\varepsilon v_F}, \tag{2.2}$$

and the function $f(\alpha) = \alpha/4$ in the perturbative regime of small α. We can re-express these results in terms of the RG equation for the dimensionless coupling α,

$$\frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3). \tag{2.3}$$

Note that α scales to small values at small energies, and this is what facilitates the transport analysis of this paper. It has been shown that $\alpha = 0$ is the only fixed point in an analysis which, in the large N limit, also remains valid for large values of α.[5,7]

We are only interested here in observables related to the electrical current, and so we will not need the explicit form of the wave-function renormalization. The current is obtained by taking a functional derivative with respect to $\mathbf{A}$, and this is protected by gauge invariance to have the same form when expressed in terms of either the bare or renormalized quantities,[14] which we will use explicitly in Eqs. (3.11) and (3.12). For two-dimensional graphene this implies that the scaling dimension of the conductivity is exactly zero and is unaffected by wave-function renormalizations. This result can also be obtained explicitly by exploring charge conservation of the system along with the related Ward identity[9] and holds to arbitrary order in perturbation theory.

We are interested here in the collision-dominated transport regime, where the characteristic energy of excitations is

T. We thus use the RG equation to scale down from some high energy cutoff scale, Λ, to a scale T. Integrating Eq. (2.3) over this regime, we obtain

$$\alpha(T) = \frac{\alpha^0}{1 + (\alpha^0/4)\ln(\Lambda/T)} \overset{T \to 0}{\sim} \frac{4}{\ln(\Lambda/T)}, \qquad (2.4)$$

where α^0 is the bare value dependent upon v_F^0. Son[5] also examined the structure of the RG flow at strong coupling in the large N limit; he found that there is a significant intermediate energy scale over which

$$\alpha(T) \sim \left(\frac{T}{\Lambda}\right)^{4/(\pi^2 N)}. \qquad (2.5)$$

Both Eqs. (2.4) and (2.5) predict a slow flow with decreasing temperature towards weak coupling. We can also use $\alpha(T)$ to obtain a T-dependent velocity,

$$v_F(T) = v_F^0 \frac{\alpha^0}{\alpha(T)} = v_F^0 \left[1 + \frac{\alpha^0}{4}\ln(\Lambda/T)\right]. \qquad (2.6)$$

We also note that the leading-order flow in α in Eq. (2.3) represents an exchange-correlation effect. Ordinary screening effects are formally higher order and can be accounted for in the random-phase approximation (RPA) by the replacement[18,19]

$$\alpha(T) \to \frac{\alpha(T)}{1 + N\pi\alpha(T)/8}. \qquad (2.7)$$

III. COLLISION-DOMINATED TRANSPORT

After initially renormalizing down to a scale T, we can now investigate the transport quantities in the renormalized theory. So all subsequent appearances of the field Ψ, the velocity v_F, and the coupling α implicitly refer to the T-dependent renormalized quantities obtained as described in Sec. II. We will not explicitly write out this T dependence.

Our formulation of the transport properties of the renormalized theory of weakly interacting massless Dirac fermions closely follows that presented in Ref. 11. This previous work considered massless Dirac fermions interacting with a weak statistical interaction due to a Chern-Simons term, and here we only need to replace the Chern-Simons term by a Coulomb interaction. The transport analysis is easiest in the real-time operator formulation with the Hamiltonian

$$H = H_0 + H_1,$$

$$H_0 = \int d\mathbf{x}[v_F \Psi_a^\dagger(-i\sigma^i\partial_i)\Psi_a], \qquad (3.1)$$

$$H_1 = \frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}$$
$$\times \Psi_a^\dagger(\mathbf{k}_2-\mathbf{q})\Psi_a(\mathbf{k}_2)V(\mathbf{q})\Psi_b^\dagger(\mathbf{k}_1+\mathbf{q})\Psi_b(\mathbf{k}_1), \qquad (3.2)$$

with the Coulomb interaction

$$V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon|\mathbf{q}|}, \qquad (3.3)$$

and $a = 1, \ldots, N$ labeling the "flavors" of fermions ($N = 4$ in graphene, accounting for two valleys and two spin projections). Even though we compute our results specifically for the Coulomb interactions [Eq. (3.3)], the formalism carries through in exactly the same manner for arbitrary isotropic two-body potentials.

The simplest formulation of the transport equations is in a basis which diagonalizes the Hamiltonian H_0. To do this, we first express Ψ in its Fourier components

$$\Psi_a(\mathbf{x},t) = \int \frac{d^2k}{(2\pi)^2}\begin{pmatrix} c_{1a}(\mathbf{k},t) \\ c_{2a}(\mathbf{k},t) \end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{x}} \qquad (3.4)$$

and then perform a unitary transformation from the Fourier mode operators (c_{1a},c_{2a}) to $(\gamma_{+a}, \gamma_{-a})$,

$$c_{1a}(k) = \frac{1}{\sqrt{2}}[\gamma_{+a}(\mathbf{k}) + \gamma_{-a}(\mathbf{k})],$$

$$c_{2a}(k) = \frac{K}{\sqrt{2}k}[\gamma_{+a}(\mathbf{k}) - \gamma_{-a}(\mathbf{k})]. \qquad (3.5)$$

We have introduced here a notational convention that we shall find quite useful in the following: as $\mathbf{k}$ is a two-dimensional momentum, we can define the complex number K by

$$K \equiv k_x + ik_y \quad \text{where} \quad \mathbf{k} \equiv (k_x, k_y) \qquad (3.6)$$

and $k = |\mathbf{k}| = |K|$. Expressing the Hamiltonian H_0 in terms of $\gamma_\pm$, we obtain the simple result,

$$H_0 = \sum_{\lambda,a}\int \frac{d^2k}{(2\pi)^2}\lambda v_F k\gamma_{\lambda a}^\dagger(\mathbf{k})\gamma_{\lambda a}(\mathbf{k}), \qquad (3.7)$$

where the sum over λ extends over $+, -$.

Let us also express the interaction Hamiltonian H_1 in terms of the $\gamma_{\lambda a}$,

$$H_1 = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4;a,b}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}$$
$$\times T_{\lambda_1\lambda_2\lambda_3\lambda_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})\gamma_{\lambda_4 b}^\dagger(\mathbf{k}_1+\mathbf{q})\gamma_{\lambda_3 a}^\dagger(\mathbf{k}_2-\mathbf{q})$$
$$\times \gamma_{\lambda_2 a}(\mathbf{k}_2)\gamma_{\lambda_1 b}(\mathbf{k}_1), \qquad (3.8)$$

where

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}) = \frac{V(\mathbf{q})}{8}\left[1 + \lambda_1\lambda_4\frac{(K_1^* + Q^*)K_1}{|\mathbf{k}_1+\mathbf{q}|k_1}\right]$$
$$\times \left[1 + \lambda_2\lambda_3\frac{(K_2^* - Q^*)K_2}{|\mathbf{k}_2-\mathbf{q}|k_2}\right]. \qquad (3.9)$$

Finally, we also express the electrical current, obtained by taking a functional derivative of the action with respect to $\mathbf{A}$, in terms of the $\gamma_\pm$. For the case of a spatially independent current (which is the only case of interest here), the result can be written as

$$\mathbf{J} = \mathbf{J}_{\mathrm{I}} + \mathbf{J}_{\mathrm{II}}, \qquad (3.10)$$

with

$$\mathbf{J}_{\mathrm{I}} = e v_F \sum_{\lambda a} \int \frac{d^2 k}{(2\pi)^2} \frac{\lambda \mathbf{k}}{k} \gamma_{\lambda a}^{\dagger}(\mathbf{k}) \gamma_{\lambda a}(\mathbf{k}) \qquad (3.11)$$

and

$$\mathbf{J}_{\mathrm{II}} = -i e v_F \int \frac{d^2 k}{(2\pi)^2} \frac{(\hat{\mathbf{z}} \times \mathbf{k})}{k} [\gamma_{+a}^{\dagger}(\mathbf{k}) \gamma_{-a}(\mathbf{k}) - \gamma_{-a}^{\dagger}(\mathbf{k}) \gamma_{+a}(\mathbf{k})], \qquad (3.12)$$

where $\hat{\mathbf{z}}$ is a unit vector orthogonal to the x, y plane. $\mathbf{J}_{\mathrm{I}}$ measures the current carried by motion of the quasiparticles and quasiholes—note the λ prefactor, indicating that these excitations have opposite charges. The operator $\mathbf{J}_{\mathrm{II}}$ creates a quasiparticle-quasihole pair [it corresponds to the so-called *Zitterbewegung* (see Ref. 1)].

As in the problems studied in Refs. 10 and 11, in a particle-hole symmetric situation a current carrying state with holes and electrons moving in opposite directions has a vanishing total momentum, and the current can decay by creation or annihilation of particle-hole pairs, without violating momentum conservation. This is the physical reason why at the particle-hole symmetric point, i.e., at vanishing deviation of the chemical potential from the Dirac point, the dc conductivity is finite even in the absence of momentum relaxing impurities. However, as we will see below, at finite deviation from particle-hole symmetry, a driving electric field always excites the system into a state with finite momentum which cannot decay. This entails an infinite dc conductivity, in accordance with the hydrodynamic analysis.[4]

Let us start by analyzing the collisionless transport equations for the quasiparticle excitations. As a first step, we define the distribution functions,

$$f_{\lambda}(\mathbf{k}, t) = \langle \gamma_{\lambda a}^{\dagger}(\mathbf{k}, t) \gamma_{\lambda a}(\mathbf{k}, t) \rangle. \qquad (3.13)$$

There is no sum over a on the right-hand side, and we assume the distribution functions to be the same for all valleys and spins. In equilibrium, i.e., in the absence of external perturbations, the distribution functions are Fermi functions,

$$f_{+}(\mathbf{k}, t) = f^0(v_F k) = \frac{1}{e^{(v_F k - \mu)/T} + 1},$$

$$f_{-}(\mathbf{k}, t) = f^0(-v_F k) = \frac{1}{e^{(-v_F k - \mu)/T} + 1}, \qquad (3.14)$$

where we temporarily allow for a finite chemical potential μ.

In principle, off-diagonal elements such as $\langle \gamma_{\pm}^{\dagger} \gamma_{\mp} \rangle$ are also created by an electric field. However, they are not needed to evaluate J_{I}, which is the part of the current that we focus on in the hydrodynamic regime, $\omega \ll k_B T$. Furthermore, the off-diagonal elements feedback to the kinetic equation of the diagonal elements only to higher order in α.

To zeroth order, in the presence of an external electric field $\mathbf{E}$, we find the simple equations,

$$\left(\frac{\partial}{\partial t} + e\mathbf{E} \cdot \frac{\partial}{\partial \mathbf{k}} \right) f_{\lambda}(\mathbf{k}, t) = 0. \qquad (3.15)$$

It is a simple matter to solve Eq. (3.15) in linear response. First we parametrize the change in f_{λ} from its equilibrium value by[20]

$$f_{\lambda}(\mathbf{k}, \omega) = 2\pi \delta(\omega) f^0(\lambda v_F k) + e \frac{\mathbf{k} \cdot \mathbf{E}(\omega)}{k} f^0(\lambda v_F k)$$

$$\times [1 - f^0(\lambda v_F k)] g_{\lambda}(k, \omega), \qquad (3.16)$$

where we have performed a Fourier transform in time to frequencies, ω, and introduced the unknown function $g_{\lambda}(k, \omega)$. At the particle-hole symmetric point ($\mu = 0$), an applied electric field generates deviations in the distribution functions having opposite sign for quasiparticles and quasiholes. Formally, this is a consequence of the driving term in Eq. (3.15) being asymmetric under $\lambda \to -\lambda$, and thus the solution has to be asymmetric as well,

$$g_{\lambda}(k, \omega) = \lambda g(k, \omega). \qquad (3.17)$$

This reflects the fact that there are an increased number of quasiholes and quasiparticles moving parallel and antiparallel to field, respectively. As quasiparticles and quasiholes have opposite charges, their electrical currents are equal, while their net momenta have opposite signs. The same argument leading to Eq. (3.17) holds true in the presence of electron-electron interactions, since the collision operator preserves the symmetry under $\lambda \to -\lambda$ (see Sec. III B).

Inserting Eq. (3.16) into Eq. (3.15), we obtain a simple solution for the function g,

$$g(k, \omega) = \frac{v_F/T}{(-i\omega + \eta)}, \qquad (3.18)$$

where η is a positive infinitesimal. Inserting this result into Eqs. (3.16) and (3.11), we obtain the conductivity

$$\sigma(\omega) = \frac{\langle J_{\mathrm{I}} \rangle}{E(\omega)} = 2N \frac{e^2 v_F}{(-i\omega + \eta)} \int \frac{d^2 k}{(2\pi)^2} \frac{k_x^2}{k^2} \left(-\frac{\partial f^0(v_F k)}{\partial k} \right)$$

$$= \frac{e^2}{h} \frac{N k_B T \ln 2}{(-i\hbar\omega + \eta)}, \qquad (3.19)$$

where, in Eq. (3.19), we have reinserted factors of $\hbar$ and k_B. Note that all factors of v_F have cancelled out: this is a consequence of the conductivity having scaling dimension $d - 2$ (where d is the spatial dimensionality) and being independent of the dynamic critical exponent z. So in this free-electron approximation, the real part of the low frequency σ is a delta function at $\omega = 0$ with weight of order $k_B T$.

Including interband transitions the real part of the conductivity becomes

$$\mathrm{Re}\,\sigma(\omega) = \frac{e^2}{h} N\pi \left[[k_B T \ln 2] \delta(\hbar\omega) + \frac{1}{8} \tanh\left(\frac{\hbar\omega}{4 k_B T} \right) \right], \qquad (3.20)$$

with high-frequency limit Re $\sigma(\omega \gg k_B T/\hbar) \rightarrow e^2 N\pi/(8h)$. In the collisionless regime this constant value remains the leading contribution to the conductivity even if one includes the electron-electron Coulomb interaction.[9,13] Next order corrections are of the form

$$\text{Re } \sigma(\omega \gg k_B T/\hbar) = \frac{e^2}{h} \frac{N\pi}{8} \{1 + \mathcal{O}[\alpha(\omega)]\}, \quad (3.21)$$

where $\alpha(\omega) \approx 4/\ln(\Lambda/\hbar\omega)$ is the renormalized frequency-dependent fine-structure constant for $k_B T \ll \hbar\omega \ll \Lambda$. For $N = 4$, this yields the result given in the upper row of Eq. (1.1). Thus, in the collisionless regime, interactions only lead to very small changes in the conductivity. In Secs. III A and III D, we will discuss the opposite, collision-dominated limit, $\hbar\omega \ll k_B T$, and determine how collisions broaden the delta function of Eq. (3.20) to a Drude peak. The latter is entirely due to the part $\mathbf{J}_{\text{I}}$ of the current, while the essential contribution of $\mathbf{J}_{\text{II}}$ is already captured by the second term in Eq. (3.20).[11]

A. Quantum Boltzmann equation

We now include collision terms on the right-hand side of Eq. (3.15). We can determine these terms by application of Fermi's golden rule[11] or by the explicit derivation presented in Appendix,

$$\left(\frac{\partial}{\partial t} + e\mathbf{E} \cdot \frac{\partial}{\partial \mathbf{k}}\right) f_\lambda(\mathbf{k},t) = -\frac{(2\pi)}{v_F} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \{\delta(k - k_1 - |\mathbf{k}+\mathbf{q}| + |\mathbf{k}_1-\mathbf{q}|) R_1(\mathbf{k},\mathbf{k}_1,\mathbf{q}) \{f_\lambda(\mathbf{k},t)f_{-\lambda}(\mathbf{k}_1,t)[1 - f_\lambda(\mathbf{k}+\mathbf{q},t)]$$
$$\times [1 - f_{-\lambda}(\mathbf{k}_1-\mathbf{q},t)] - [1 - f_\lambda(\mathbf{k},t)][1 - f_{-\lambda}(\mathbf{k}_1,t)] f_\lambda(\mathbf{k}+\mathbf{q},t) f_{-\lambda}(\mathbf{k}_1-\mathbf{q},t)\}$$
$$\times \delta(k + k_1 - |\mathbf{k}+\mathbf{q}| - |\mathbf{k}_1-\mathbf{q}|) R_2(\mathbf{k},\mathbf{k}_1,\mathbf{q}) \{f_\lambda(\mathbf{k},t) f_\lambda(\mathbf{k}_1,t)[1 - f_\lambda(\mathbf{k}+\mathbf{q},t)]$$
$$\times [1 - f_\lambda(\mathbf{k}_1-\mathbf{q},t)] - [1 - f_\lambda(\mathbf{k},t)][1 - f_\lambda(\mathbf{k}_1,t)] f_\lambda(\mathbf{k}+\mathbf{q},t) f_\lambda(\mathbf{k}_1-\mathbf{q},t)\}\}, \quad (3.22)$$

where

$$R_1(\mathbf{k},\mathbf{k}_1,\mathbf{q}) = 4(|T_{+--+}(\mathbf{k},\mathbf{k}_1,\mathbf{q}) - T_{+-+-}(\mathbf{k},\mathbf{k}_1,-\mathbf{k}-\mathbf{q}+\mathbf{k}_1)|^2 + (N-1)|T_{+--+}(\mathbf{k},\mathbf{k}_1,\mathbf{q})|^2 + (N-1)|T_{+-+-}(\mathbf{k},\mathbf{k}_1,-\mathbf{k}-\mathbf{q}+\mathbf{k}_1)|^2),$$

$$(3.23)$$

$$R_2(\mathbf{k},\mathbf{k}_1,\mathbf{q}) = 4\left(\frac{1}{2}|T_{+--+}(\mathbf{k},\mathbf{k}_1,\mathbf{q}) - T_{++++}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_1-\mathbf{k}-\mathbf{q})|^2 + (N-1)|T_{++++}(\mathbf{k},\mathbf{k}_1,\mathbf{q})|^2\right), \quad (3.23)$$

which are illustrated in Fig. 3. The terms proportional to R_1 represent collisions between oppositely charged particles, while those proportional to R_2 are collisions between like charges. Other processes where a particle-hole pair is created turn out to have vanishing phase space: it is not possible to fulfill momentum and energy conservation at the same time. This is a peculiarity of the linear dispersion, i.e., $\varepsilon_k = v_F k$ (see also Refs. 10 and 11).

We now proceed to the linearization of Eq. (3.23) by inserting parametrization (3.16) and find

$$\frac{[-i\omega g_\lambda(k,\omega) - \lambda v_F/T]}{(e^{v_F k/T} + 1)(e^{-v_F k/T} + 1)} \frac{\mathbf{k}}{k} = -\frac{(2\pi)}{v_F} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left\{ \frac{\delta(k - k_1 - |\mathbf{k}+\mathbf{q}| + |\mathbf{k}_1-\mathbf{q}|) R_1(\mathbf{k},\mathbf{k}_1,\mathbf{q})}{(e^{-v_F k/T} + 1)(e^{v_F k_1/T} + 1)(e^{v_F |\mathbf{k}+\mathbf{q}|/T} + 1)(e^{-v_F |\mathbf{k}_1-\mathbf{q}|/T} + 1)} \right.$$
$$\times \left[\frac{\mathbf{k}}{k} g_\lambda(k,\omega) + \frac{\mathbf{k}_1}{k_1} g_{-\lambda}(k_1,\omega) - \frac{(\mathbf{k}+\mathbf{q})}{|\mathbf{k}+\mathbf{q}|} g_\lambda(|\mathbf{k}+\mathbf{q}|,\omega) - \frac{(\mathbf{k}_1-\mathbf{q})}{|\mathbf{k}_1-\mathbf{q}|} g_{-\lambda}(|\mathbf{k}_1-\mathbf{q}|,\omega) \right]$$
$$+ \frac{\delta(k + k_1 - |\mathbf{k}+\mathbf{q}| - |\mathbf{k}_1-\mathbf{q}|) R_2(\mathbf{k},\mathbf{k}_1,\mathbf{q})}{(e^{-v_F k/T} + 1)(e^{-v_F k_1/T} + 1)(e^{v_F |\mathbf{k}+\mathbf{q}|/T} + 1)(e^{v_F |\mathbf{k}_1-\mathbf{q}|/T} + 1)}$$
$$\left. \times \left[\frac{\mathbf{k}}{k} g_\lambda(k,\omega) + \frac{\mathbf{k}_1}{k_1} g_\lambda(k_1,\omega) - \frac{(\mathbf{k}+\mathbf{q})}{|\mathbf{k}+\mathbf{q}|} g_\lambda(|\mathbf{k}+\mathbf{q}|,\omega) - \frac{(\mathbf{k}_1-\mathbf{q})}{|\mathbf{k}_1-\mathbf{q}|} g_\lambda(|\mathbf{k}_1-\mathbf{q}|,\omega) \right] \right\}. \quad (3.24)$$
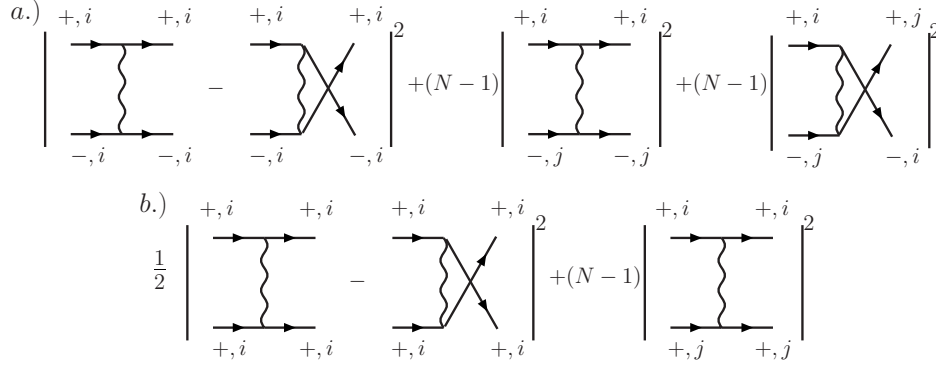
FIG. 3. Illustration of the golden rule diagrams entering the collision term. The diagrams (a) describe scattering of oppositely charged particles corresponding to the term R_1, while the diagrams (b) describe scattering of like particles corresponding to the term R_2. Note that the vertex preserves the flavor $a=i,j$ but not the particle/hole nature $\lambda=\pm$. The factor $1/2$ of the first diagram accounts for the symmetry factor associated with having two indistinguishable particles in the final state.

The remainder of this paper is focused on the solution of the linearized transport equation in Eq. (3.24) for the function g. It is useful at this point to recall some crucial mathematical properties of such transport equations, reviewed, e.g., by Ziman[21] and Arnold *et al.*[20] We can view the right-hand side of Eq. (3.24) as a linear operator, the so-called collision operator $\mathcal{C}$, acting on the function $(\mathbf{k}/k)g(k)$; we drop the implicit ω dependence because $\mathcal{C}$ is independent of ω. A key property of $\mathcal{C}$ is that it is Hermitian with respect to the natural inner product,

$$\langle g_1|g_2\rangle \equiv \sum_\lambda \int \frac{d^2k}{(2\pi)^2} g_{1,\lambda}(k)g_{2,\lambda}(k). \qquad (3.25)$$

This Hermiticity follows[20] from symmetry properties of R_1 and R_2 under exchanges between incoming and outgoing mo-

menta, which are very similar to those used in establishing Boltzmann's H theorem.

Related to the above properties of the collision operator, we can introduce a functional $\mathcal{Q}[g]$, such that Eq. (3.24) is equivalent to finding its stationary point,

$$\frac{\delta\mathcal{Q}[g]}{\delta g} = 0. \qquad (3.26)$$

Specializing to the particle-hole symmetric case [cf. Eq. (3.17)], the explicit form of the functional is

$$
\begin{aligned}
\mathcal{Q}[g] = \frac{(2\pi)}{8v_F} \int & \frac{d^2k}{(2\pi)^2}\frac{d^2k_1}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\Bigg\{ \frac{\delta(k-k_1-|\mathbf{k}+\mathbf{q}|+|\mathbf{k}_1-\mathbf{q}|)R_1(\mathbf{k},\mathbf{k}_1,\mathbf{q})}{(e^{-v_Fk/T}+1)(e^{v_Fk_1/T}+1)(e^{v_F|\mathbf{k}+\mathbf{q}|/T}+1)(e^{-v_F|\mathbf{k}_1-\mathbf{q}|/T}+1)} \\
& \times \left[\frac{\mathbf{k}}{k}g(k,\omega) - \frac{\mathbf{k}_1}{k_1}g(k_1,\omega) - \frac{(\mathbf{k}+\mathbf{q})}{|\mathbf{k}+\mathbf{q}|}g(|\mathbf{k}+\mathbf{q}|,\omega) + \frac{(\mathbf{k}_1-\mathbf{q})}{|\mathbf{k}_1-\mathbf{q}|}g(|\mathbf{k}_1-\mathbf{q}|,\omega) \right]^2 \\
+ & \frac{\delta(k+k_1-|\mathbf{k}+\mathbf{q}|-|\mathbf{k}_1-\mathbf{q}|)R_2(\mathbf{k},\mathbf{k}_1,\mathbf{q})}{(e^{-v_Fk/T}+1)(e^{-v_Fk_1/T}+1)(e^{v_F|\mathbf{k}+\mathbf{q}|/T}+1)(e^{v_F|\mathbf{k}_1-\mathbf{q}|/T}+1)} \\
& \times \left[\frac{\mathbf{k}}{k}g(k,\omega) + \frac{\mathbf{k}_1}{k_1}g(k_1,\omega) - \frac{(\mathbf{k}+\mathbf{q})}{|\mathbf{k}+\mathbf{q}|}g(|\mathbf{k}+\mathbf{q}|,\omega) - \frac{(\mathbf{k}_1-\mathbf{q})}{|\mathbf{k}_1-\mathbf{q}|}g(|\mathbf{k}_1-\mathbf{q}|,\omega) \right]^2 \Bigg\} + \int \frac{d^2k}{(2\pi)^2}\frac{g(k,\omega)[-i\omega g(k,\omega)/2 - v_F/T]}{(e^{v_Fk/T}+1)(e^{-v_Fk/T}+1)}.
\end{aligned}
$$

$$(3.27)$$

B. Translational invariance and momentum conservation

The translational invariance of the system immediately implies the presence of a zero mode of the operator $\mathcal{C}$, which corresponds to the shift of the distribution functions arising from changing to a linearly moving reference frame. The

corresponding deviation $g_\lambda(k,\omega)$ has the form

$$g_\lambda(k,\omega) = \chi(\omega)\psi_{tr}(k) \equiv \chi(\omega)k, \qquad (3.28)$$

which is easily seen to annihilate the right-hand side of Eq. (3.24) due to momentum conservation. Note that this zero

mode of the Boltzmann operator is orthogonal to any modes of the form (3.17) which are the only ones that can be excited by an electric field at particle-hole symmetry for the reasons already explained. This again reflects the fact that current and momentum are independent of each other at this special point. However, away from the Dirac point or if a thermal gradient is applied instead of an electric field, the zero mode ψ_{tr} will be excited by the driving field which leads to a diverging dc response in clean systems. This will be discussed in more detail in a forthcoming publication.

For the following we restrict ourselves to the electrical conductivity at the particle-hole symmetric point where the above zero mode is not excited and thus irrelevant. In order to obtain a better understanding for the relaxation processes, it is interesting to consider a similar mode,

$$g_\lambda(k, \omega) = \lambda \widetilde{\chi}(\omega) \psi_{\text{tr}}(k) \equiv \widetilde{\chi}(\omega) \lambda k, \qquad (3.29)$$

which, however, satisfies the constraint [Eq. (3.17)]. This mode also annihilates R_2 in Eq. (3.27) both for electrons and for holes. Consequently, the scattering of like charges cannot relax this mode. This is not true for R_1 which implies that this mode does relax but only due to scattering of electrons off holes and vice versa. This illustrates again the importance of there being particles and holes for a finite conductivity to be possible.

C. Collinear limit

In the previous analysis of a quantum Boltzmann equation for massless Dirac fermions in two dimensions,[11] it was noted that the phase space for scattering of particles was logarithmically divergent in the collinear limit. For the interaction considered in that paper, the collinear scattering cross section vanished, and so this singular phase-space density had no important consequences. The collinear scattering does not vanish for the present Coulomb interaction, and so we need to consider this logarithmic divergence seriously.

The physical origin of the divergent collinear scattering is related to the linear dispersion which implies that quasiparticles or quasiholes moving in the same direction share the same group velocity, independent of their energies. This leads to a diverging duration of collisions of nearly collinear particles, which is enhanced due to the low space dimensionality. To the extent that collinear scattering is very strong and considering frequencies much smaller than the inelastic-scattering rate, we may expect that quasiparticles and quasiholes that move in the same direction in the plane will establish a pseudoequilibrium characterized by an effective chemical potential and an effective temperature which, however, depends on the direction of motion.

In linear response the deviations of these effective parameters from the equilibrium values μ and T have to vary with $\mathbf{k}/k \cdot \mathbf{E}$ for symmetry reasons. Further, the effective temperature shift is easily shown to be identical to the mode ψ_{tr} discussed above, and it is thus ruled out at the particle-hole symmetric point $\mu = 0$. The remaining dominant mode of the function g will correspond to an effective shift in chemical potential which translates into

$$g_\lambda(k, \omega) = \chi(\omega) \psi_\mu(k) \equiv \frac{v_F}{T^2} \lambda \chi(\omega), \qquad (3.30)$$

where the prefactor has been chosen so as to make $\chi(\omega)$ dimensionless. With this ansatz, which will be confirmed below, it simply remains to determine the prefactor $\chi(\omega)$, yielding the leading term in the nonequilibrium distribution. Note that the effective chemical-potential shift ranges between $\pm \chi \hbar v_F eE/T$ depending on the direction of motion. Comparing this to the temperature allows us to estimate the threshold electric-field strength, $eE_{\text{lin}} = T^2/\hbar v_F$, below which nonlinear effects should remain small.

Let us now review in more detail how the above physical picture arises in the formalism of the Boltzmann equation. The occurrence of a logarithmic divergence can be seen by allowing the incoming and outgoing momenta to be nearly collinear. Without loss of generality, we choose $\mathbf{k} = (k, 0)$, with $k > 0$. Also, we write $\mathbf{k}_1 = (k_1, k_\perp)$ and $\mathbf{q} = (q, q_\perp)$ with $k_\perp$ and $q_\perp$ small. The divergence in the phase-space density of the collision term proportional to R_2 occurs when $k_1 > 0$, $k + q > 0$, and $k_1 - q > 0$. Likewise, for scattering of oppositely charged particles a divergence occurs when their k vectors are anticollinear which ensures collinear group velocities since $\mathbf{v}_\mathbf{k} = \lambda v_F \mathbf{k}/k$. In this regime, the argument of the energy conservation delta function of the particle-particle scattering term can be written as

$$k + k_1 - |\mathbf{k} + \mathbf{q}| - |\mathbf{k}_1 - \mathbf{q}|$$

$$\approx \frac{k_\perp^2}{2k_1} - \frac{q_\perp^2}{2(k+q)} - \frac{(k_\perp - q_\perp)^2}{2(k_1 - q)}$$

$$\equiv -\frac{(k + k_1)}{2(k+q)(k_1 - q)}(q_\perp - \zeta_1 k_\perp)(q_\perp - \zeta_2 k_\perp),$$

$$(3.31)$$

where $\zeta_{1,2}$ depend upon k, k_1, and q and are the roots of a quadratic equation which are defined by the expressions above. Then, the phase-space density for the R_2 term is proportional to

$$\int dk_\perp dq_\perp \, \delta(k + k_1 - |\mathbf{k} + \mathbf{q}| - |\mathbf{k}_1 - \mathbf{q}|)$$

$$= \int \frac{dk_\perp}{|k_\perp|} \frac{4(k+q)(k_1 - q)}{(k + k_1)|\zeta_1 - \zeta_2|}$$

$$= 2\sqrt{\frac{k_1(k+q)(k_1 - q)}{k}} \int \frac{dk_\perp}{|k_\perp|}. \qquad (3.32)$$

The logarithmic divergence as $k_\perp \rightarrow 0$ is now evident. This divergence is clearly a consequence of the linear dispersion of the fermions, and the above analysis also makes it clear that it is special to two dimensions. As discussed in Ref. [20] for a similar divergence in a different problem, we expect that this divergence is cutoff by higher-order self-energy corrections to the fermions. Such self-energy corrections appear at order α in the perturbation theory, and so the important range of the $k_\perp$ integral is between T/v_F and $T\alpha/v_F$. So we may approximate[20]

$$\int \frac{dk_\perp}{|k_\perp|} \approx 2 \ln(1/\alpha), \tag{3.33}$$

and set $k_\perp = q_\perp = 0$ elsewhere to obtain the leading contribution to the collision integral in the limit $\alpha \to 0$. Proceeding in this manner, the part of $\mathcal{C}$ on the right-hand side of Eq. (3.24) proportional to R_2, which we denote $\mathcal{C}_2$, becomes

$$\mathcal{C}_2[g] \approx -\frac{\ln(1/\alpha)}{2\pi^3 v_F} \frac{\mathbf{k}}{k} \int_0^\infty dk_1 \int_{-k}^{k_1} \frac{dq}{q^2} \sqrt{\frac{k_1(k+q)(k_1-q)}{k}} \frac{R_2[g(k,\omega) + g(k_1,\omega) - g(k+q,\omega) - g(k_1-q,\omega)]}{(e^{-v_F k/T} + 1)(e^{-v_F k_1/T} + 1)(e^{v_F(k+q)/T} + 1)(e^{v_F(k_1-q)/T} + 1)}. \tag{3.34}$$

Consonant with our discussion earlier in this section, a key property of the above expression for $\mathcal{C}_2$ was noted by Kashuba:[16] the function $g = $ constant is an eigenvector of $\mathcal{C}_2$ with zero eigenvalue. The same is also easily seen to apply to the portion $\mathcal{C}_1$ of $\mathcal{C}$ which is proportional to R_1. Indeed, this is just the direction-specific chemical-potential shift in Eq. (3.30), which naturally is a zero mode for collinear scattering, since it maintains a pseudoequilibrium among particles moving in the same direction.

Going beyond the collinear limit, we conclude that there is an eigenvalue of $\mathcal{C}$ which is not proportional to $\ln(1/\alpha)$ in the limit of small α; the corresponding eigenvector is given by a constant $g(k)$ up to corrections of order $[\ln(1/\alpha)]^{-1}$.

The solution of the Boltzmann equation in Eq. (3.24) requires that we obtain the operator $\mathcal{C}^{-1}$, and the results above allow us to constrain its form in the limit $\ln(1/\alpha) \geqslant 1$. Let $|\mu\rangle$ be the eigenvectors of $\mathcal{C}$ with eigenvalues λ_μ. Then

$$\mathcal{C}^{-1} = \sum_\mu \frac{|\mu\rangle\langle\mu|}{\lambda_\mu}, \tag{3.35}$$

and in the limit of large $\ln(1/\alpha)$, $\mathcal{C}^{-1}$ is dominated[16] by the eigenvector whose eigenvalue is not proportional to $\ln(1/\alpha)$. Note that it is quite remarkable that in this limit we can solve the Boltzmann equation essentially exactly.

D. Results

From the reasoning in Sec. III C, we conclude that up to corrections of order $[\ln(1/\alpha)]^{-1}$, we can choose g to be of the form

$$g(k,\omega) \approx \frac{v_F}{T^2} C(\omega). \tag{3.36}$$

We insert this parametrization into the functional $\mathcal{Q}[g]$ in Eq. (3.27); the solution of the stationarity condition in Eq. (3.26) is then equivalent to requiring the vanishing of the derivative with respect to C. We numerically evaluated the integrals in Eq. (3.27) using an elliptic coordinate system to solve the energy conservation constraint[11] and obtained

$$\mathcal{Q}[g] = \frac{1}{T} \frac{\ln 2}{4\pi} \left[\kappa \alpha^2 C^2(\omega) - 2C(\omega) - i\left(\frac{\omega}{T}\right) C^2(\omega) \right],$$

with $\kappa = 3.646$ for the physical case $N = 4$. From the stationarity condition we then obtain

$$C(\omega) = \frac{1}{-i(\omega/T) + \kappa\alpha^2}. \tag{3.37}$$

The conductivity can be obtained from $C(\omega)$ by combining Eqs. (3.11), (3.16), and (3.36),

$$\sigma(\omega) = \frac{e^2}{h} \frac{N k_B T \ln 2}{-i\hbar\omega + \kappa k_B T \alpha^2}, \tag{3.38}$$

where we have reinserted factors of $\hbar$ and k_B. Note that the conductivity depends only upon $\alpha(T)$, while all other factors of $v_F(T)$ cancel. Note also the connection to the free particle result in Eq. (3.19)—the only difference is that the infinitesimal η has been replaced by $\hbar$ times the inelastic relaxation rate $\hbar\tau^{-1} = \kappa k_B T \alpha^2$.

IV. CONCLUSIONS

We conclude by briefly noting the conditions under which our main results for the conductivity in Eqs. (3.38), (2.4), and (2.5) may be observed in transport measurements. The key requirement is that $k_B T \alpha^2$ be the largest infrared energy scale which quenches the ideal Dirac fermion behavior. Thus, the sample size should be larger than the inelastic-scattering length $\ell_{ee} \approx \hbar v_F / (k_B T \alpha^2)$. Similarly, the elastic mean-free path from impurity scattering should be larger than ℓ_{ee} too. Present experiments do not seem to have reached this regime of purity yet. However, we hope that the clean limit can be approached in future experiments, e.g., on suspended graphene sheets for which our results should be relevant in a range of intermediate temperatures where disorder effects are still subdominant.

Note that the existence of a finite conductivity only depends on particle-hole symmetry. The essential features of our results remain unchanged even if a small gap opens at the Dirac point, as long as the gap is significantly smaller than temperature, and as long as the curvature of the energy dispersion is negligible over the range of thermal energies.

Since particle-hole symmetry is required for our theory to apply, the bias voltage should be smaller than $k_B T \alpha^2$. Note that even though we have neglected the scattering from phonons in the present work, it is by such processes that the (small) Joule heat produced in the dissipative conduction process is eventually transmitted to lattice vibrational modes preventing a heating of the electron system.

It is possible to extend our analysis to include all the additional perturbations noted in the previous paragraph, with a treatment of disorder effects following that of Ref. 22. When these perturbations are weak (compared to $k_B T \alpha^2$), then in the collision-dominated regime, a general hydrodynamic analysis is possible: this was presented recently in Ref. 4. Also, in this regime the analysis of the Boltzmann equation greatly simplifies if the interactions are weak enough to ensure a strong logarithmic divergence in the collinear channel. The latter establishes pseudoequilibrium along different directions if the inelastic-scattering time remains the shortest relevant time scale in the problem. Otherwise, a full analysis of the modified quantum Boltzmann equation is required. These aspects will be discussed in future work.

APPENDIX: DERIVATION OF THE QUANTUM KINETIC EQUATION

An alternative derivation of the quantum kinetic equation can be carried out in the framework of closed time contour ordered perturbation theory, as explicated in Chap. 9 of Ref. 23. The problem we consider falls into the generic class of a system describing particles interacting via a distance-dependent density-density interaction. Thus the Hamiltonian is of the form

$$H = H_0 + \frac{1}{2}\int d^2\mathbf{r}\, d^2\mathbf{r}'\, V(\mathbf{r},\mathbf{r}')\rho(\mathbf{r})\rho(\mathbf{r}'), \tag{A1}$$

where $\rho(\mathbf{r})$ denotes the particle density at spatial point $\mathbf{r}$. The following considerations are completely generic and apply to any system which falls into the class of Hamiltonians presented in Eq. (A1). The starting point of our discussion is given by Eqs. (9)–(7a) of Ref. 23, which has to be generalized to incorporate a possible matrix structure of the Green's function (in our case the Green's function lives in spinor space within a structure due to the N spin and valley species and thus has a $2N \times 2N$ structure),

$$[\partial_T - \nabla_{\mathbf{R}} U(\mathbf{R},T)\nabla_{\mathbf{k}}]\mathbf{G}^<(\mathbf{k},\omega;\mathbf{R},T)$$
$$= -\mathbf{G}^<(\mathbf{k},\omega;\mathbf{R},T)\mathbf{\Sigma}^>(\mathbf{k},\omega;\mathbf{R},T)$$
$$+ \mathbf{G}^>(\mathbf{k},\omega;\mathbf{R},T)\mathbf{\Sigma}^<(\mathbf{k},\omega;\mathbf{R},T), \tag{A2}$$

where

$$\mathbf{\Sigma}_{\alpha\beta}^{>,<}(\mathbf{r},t;\mathbf{R},T)$$
$$\approx -i^2 \int d\overline{\mathbf{R}}\, d\overline{\mathbf{r}}\, V(\mathbf{R}+\mathbf{r}/2-\overline{\mathbf{R}}-\overline{\mathbf{r}}/2) V(\mathbf{R}-\mathbf{r}/2-\overline{\mathbf{R}}+\overline{\mathbf{r}}/2)$$
$$\times \mathbf{G}_{\gamma\delta}^{<,>}(-\overline{\mathbf{r}},-t;R,T)[\mathbf{G}_{\alpha\beta}^{>,<}(\mathbf{r},t;R,T)\mathbf{G}_{\delta\gamma}^{>,<}(\overline{\mathbf{r}},t;R,T)$$
$$- \mathbf{G}_{\alpha\gamma}^{>,<}(\mathbf{R}+\overline{\mathbf{r}}/2-\overline{\mathbf{R}}+\mathbf{r}/2,t;R,T)$$
$$\times \mathbf{G}_{\delta\beta}^{>,<}(\mathbf{R}+\overline{\mathbf{r}}/2-\overline{\mathbf{R}}+\mathbf{r}/2,t;R,T)] \tag{A3}$$

in the Born approximation (note that double indices are summed over). The corresponding self-energy diagrams are the RPA-type contribution and the maximally crossed diagram (see also Ref. 23). In a next step, following the treatment of Kadanoff and Baym,[23] we find that the Fourier transform with respect to the relative coordinates (which corresponds to the mixed Wigner transform) of Eq. (A3) reads (note that in the following we drop the dependence on the center-of-mass coordinate $\mathbf{R}$)

$$\mathbf{\Sigma}_{\alpha\beta}^{>,<}(\mathbf{k},\omega;T)$$
$$= \int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d\omega_1}{2\pi}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\frac{d\omega_2}{2\pi}\frac{d^2\mathbf{k_3}}{(2\pi)^2}\frac{d\omega_3}{2\pi}(2\pi)^3$$
$$\times \delta(\mathbf{k}+\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\delta(\omega+\omega_1-\omega_2-\omega_3)$$
$$\times [V(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_2})\mathbf{G}_{\gamma\delta}^{<,>}(\mathbf{k_1},\omega_1)\mathbf{G}_{\alpha\beta}^{>,<}(\mathbf{k_2},\omega_2)$$
$$\times \mathbf{G}_{\delta\gamma}^{>,<}(\mathbf{k_3},\omega_3) - V(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_3})$$
$$\times \mathbf{G}_{\gamma\delta}^{<,>}(\mathbf{k},\omega_1)\mathbf{G}_{\alpha\gamma}^{>,<}(\mathbf{k_2},\omega_2)\mathbf{G}_{\delta\beta}^{>,<}(\mathbf{k_3},\omega_3)]. \tag{A4}$$

Until now all the formulas are completely generic and not specific to graphene. In order to make connection to the problem of graphene we note that the Green's function of the spinors Ψ is related to the Green's function of the γ through

$$\mathbf{G}^{<,>}(\mathbf{k},\omega) = U_{\mathbf{k}}^{-1} g^{<,>}(\mathbf{k},\omega) U_{\mathbf{k}},$$

where the unitary matrix $U_{\mathbf{k}}^{-1}$ according to Eq. (3.5) is given by

$$U_{\mathbf{k}}^{-1} = \frac{1}{\sqrt{2}k}\begin{pmatrix} k & k \\ K & -K \end{pmatrix}.$$

Furthermore we note that the summation over spin and valley indices only affects the RPA-type diagram, which thus receives a prefactor N and the resulting matrix equation is an equation, whose indices only carry over the 2×2 matrix in spinor space. This allows us to rewrite Eq. (A4) as

$$\mathbf{\Sigma}_{\alpha\beta}^{>,<}(\mathbf{k},\omega;T) = \int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d\omega_1}{2\pi}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\frac{d\omega_2}{2\pi}\frac{d^2\mathbf{k_3}}{(2\pi)^2}\frac{d\omega_3}{2\pi}(2\pi)^3\delta(\mathbf{k}+\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\times\delta(\omega+\omega_1-\omega_2-\omega_3)\times\{NV(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_2})$$

$$\times[U_{\mathbf{k_1}}^{-1}g^{<,>}(\mathbf{k_1},\omega_1)U_{\mathbf{k_1}}]_{\gamma\delta}[U_{\mathbf{k_2}}^{-1}g^{>,<}(\mathbf{k_2},\omega_2)U_{\mathbf{k_2}}]_{\alpha\beta}[U_{\mathbf{k_3}}^{-1}g^{>,<}(\mathbf{k_3},\omega_3)U_{\mathbf{k_3}}]_{\delta\gamma} - V(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_3})$$

$$\times[U_{\mathbf{k_1}}^{-1}g^{<,>}(\mathbf{k_1},\omega_1)U_{\mathbf{k_1}}]_{\gamma\delta}[U_{\mathbf{k_2}}^{-1}g^{>,<}(\mathbf{k_2},\omega_2)U_{\mathbf{k_2}}]_{\alpha\gamma}[U_{\mathbf{k_3}}^{-1}g^{>,<}(\mathbf{k_3},\omega_3)U_{\mathbf{k_3}}]_{\delta\beta}\}. \tag{A5}$$

Accounting for the fact that the operators γ describe sharp quasiparticles the lesser and greater Green's functions are given by

$$g_{\lambda\lambda'}^{<}(\mathbf{k},\omega;T) = 2\pi\delta[\omega - \epsilon_\lambda(\mathbf{k},T)]f_\lambda(\mathbf{k},T)\delta_{\lambda,\lambda'}$$

and

$$g_{\lambda\lambda'}^{>}(\mathbf{k},\omega;T) = 2\pi\delta[\omega - \epsilon_\lambda(\mathbf{k},T)][1 - f_\lambda(\mathbf{k},T)]\delta_{\lambda,\lambda'},$$

where we assumed the distribution function of the quasiparticles to have no off-diagonal components, which is justified to linear order in the potential gradient. We can formulate the kinetic equation for the diagonal part of the distribution function as

$$[\partial_T - \nabla_{\mathbf{R}}U(\mathbf{R},T)\nabla_{\mathbf{k}}]f_\mu(\mathbf{k},T)$$
$$= -f_\mu(\mathbf{k},T)\{U_{\mathbf{k}}\mathbf{\Sigma}^{>}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});T]U_{\mathbf{k}}^{-1}\}_{\mu\mu}$$
$$+ [1 - f_\mu(\mathbf{k},T)]\{U_{\mathbf{k}}\mathbf{\Sigma}^{<}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});T]U_{\mathbf{k}}^{-1}\}_{\mu\mu} \tag{A6}$$

or equivalently

$$[\partial_T - \nabla_{\mathbf{R}}U(\mathbf{R},T)\nabla_{\mathbf{k}}]f_\mu(\mathbf{k},T)$$
$$= -f_\mu(\mathbf{k},T)\sigma_{\mu\mu}^{>}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});T]$$
$$+ [1 - f_\mu(\mathbf{k},T)]\sigma_{\mu\mu}^{<}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});T]. \tag{A7}$$

Exploiting the form of the lesser and greater Green's functions we can rewrite the self-energies as (note that μ, in contrast to the other double indices, is not summed over here and subsequently)

$$\sigma_{\mu\mu}^{>}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});T] = \int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d\omega_1}{2\pi}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\frac{d\omega_2}{2\pi}\frac{d^2\mathbf{k_3}}{(2\pi)^2}\frac{d\omega_3}{2\pi}(2\pi)^3\delta(\mathbf{k}+\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})$$

$$\times \delta[\epsilon_\mu(\mathbf{k})+\omega_1-\omega_2-\omega_3]\times\{NV(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_2})(2\pi)^3\delta[\omega_1-\epsilon_\lambda(\mathbf{k_1})]\delta[\omega_2-\epsilon_{\lambda_1}(\mathbf{k_2})]\delta[\omega_3-\epsilon_{\lambda_2}(\mathbf{k_3})]$$

$$\times M_{\lambda_2\lambda}(\mathbf{k_3},\mathbf{k_1})M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_3})M_{\mu\lambda_1}(\mathbf{k},\mathbf{k_2})M_{\lambda_1\mu}(\mathbf{k_2},\mathbf{k})f_\lambda(\mathbf{k_1},T)[1-f_{\lambda_1}(\mathbf{k_2},T)][1-f_{\lambda_2}(\mathbf{k_3},T)] - V(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_3})$$

$$\times(2\pi)^3\delta[\omega_1-\epsilon_\lambda(\mathbf{k_1})]\delta[\omega_2-\epsilon_{\lambda_1}(\mathbf{k_2})]\delta[\omega_3-\epsilon_{\lambda_2}(\mathbf{k_3})]$$

$$\times M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_3})M_{\lambda_1\lambda}(\mathbf{k_2},\mathbf{k_1})M_{\mu\lambda_1}(\mathbf{k},\mathbf{k_2})T_{\lambda_2\mu}(\mathbf{k_3},\mathbf{k})f_\lambda(\mathbf{k_1},T)[1-f_{\lambda_1}(\mathbf{k_2},T)][1-f_{\lambda_2}(\mathbf{k_3},T)]\} \tag{A8}$$

and

$$\sigma_{\mu\mu}^{<}[\mathbf{k},\omega = \epsilon_\mu(\mathbf{k});\mathbf{R},T] = \int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d\omega_1}{2\pi}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\frac{d\omega_2}{2\pi}\frac{d^2\mathbf{k_3}}{(2\pi)^2}\frac{d\omega_3}{2\pi}(2\pi)^3\delta(\mathbf{k}+\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})$$

$$\times \delta[\epsilon_\mu(\mathbf{k})+\omega_1-\omega_2-\omega_3]\times\{NV(\mathbf{k}-\mathbf{k_2})V(\mathbf{k_1}-\mathbf{k_2})(2\pi)^3\delta[\omega_1-\epsilon_\lambda(\mathbf{k_1})]\delta[\omega_2-\epsilon_{\lambda_1}(\mathbf{k_2})]\delta[\omega_3-\epsilon_{\lambda_2}(\mathbf{k_3})]$$

$$\times M_{\lambda_2\lambda}(\mathbf{k_3},\mathbf{k_1})M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_3})M_{\mu\lambda_1}(\mathbf{k},\mathbf{k_2})M_{\lambda_1\mu}(\mathbf{k_2},\mathbf{k})[1-f_\lambda(\mathbf{k_1},T)]f_{\lambda_1}(\mathbf{k_2},T)f_{\lambda_2}(\mathbf{k_3},T) - V(\mathbf{k}-\mathbf{k_2})V(\mathbf{k}-\mathbf{k_3})$$

$$\times(2\pi)^3\delta[\omega_1-\epsilon_\lambda(\mathbf{k_1})]\delta[\omega_2-\epsilon_{\lambda_1}(\mathbf{k_2})]\delta[\omega_3-\epsilon_{\lambda_2}(\mathbf{k_3})]$$

$$\times M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_3})M_{\lambda_1\lambda}(\mathbf{k_2},\mathbf{k_1})M_{\mu\lambda_1}(\mathbf{k},\mathbf{k_2})M_{\lambda_2\mu}(\mathbf{k_3},\mathbf{k})[1-f_\lambda(\mathbf{k_1},T)]f_{\lambda_1}(\mathbf{k_2},T)f_{\lambda_2}(\mathbf{k_3},T)\}, \tag{A9}$$

where the shorthand notation

$$M_{\lambda\lambda_1}(\mathbf{k},\mathbf{k_1}) = [U_{\mathbf{k}}U_{\mathbf{k_1}}^{-1}]_{\lambda\lambda_1} = \frac{1}{2}\begin{pmatrix} 1+\dfrac{K^*K_1}{kk_1} & 1-\dfrac{K^*K_1}{kk_1} \\ 1-\dfrac{K^*K_1}{kk_1} & 1+\dfrac{K^*K_1}{kk_1} \end{pmatrix}_{\lambda\lambda_1} = \frac{1}{2}\left(1+\lambda\lambda_1\frac{K^*K_1}{kk_1}\right)$$

was introduced. The connection with the matrix elements defined in Eq. (3.9) can be easily established and reads

$$T_{\lambda\lambda_1\lambda_2\lambda_3}(\mathbf{k},\mathbf{k_1},\mathbf{q}) = \frac{1}{2}V(-\mathbf{q})M_{\lambda\lambda_3}(\mathbf{k}+\mathbf{q},\mathbf{k})M_{\lambda_1\lambda_2}(\mathbf{k_1}-\mathbf{q},\mathbf{k_1}).$$ (A10)

Using Eq. (A7) and preforming a sequence of transformations finally yields

$$\begin{aligned}[\partial_T - \nabla_{\mathbf{R}}U(\mathbf{R},T)\nabla_{\mathbf{k}}]f_\mu(\mathbf{k},T) = &\frac{2\pi}{v_F}\int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d^2\mathbf{q}}{(2\pi)^2}\delta(\mu k + \lambda k_1 - \lambda_1|\mathbf{k}+\mathbf{q}| - \lambda_2|\mathbf{k_1}-\mathbf{q}|)\\
&\times [NV(-\mathbf{q})V(-\mathbf{q})M_{\lambda_2\lambda}(\mathbf{k_1}-\mathbf{q},\mathbf{k_1})M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_1}-\mathbf{q})M_{\mu\lambda_1}(\mathbf{k},\mathbf{q}+\mathbf{k})M_{\lambda_1\mu}(\mathbf{q}+\mathbf{k},\mathbf{k})\\
&- V(-\mathbf{q})V(\mathbf{k}-\mathbf{k_1}+\mathbf{q})M_{\lambda\lambda_2}(\mathbf{k_1},\mathbf{k_1}-\mathbf{q})M_{\lambda_1\lambda}(\mathbf{q}+\mathbf{k},\mathbf{k_1})M_{\mu\lambda_1}(\mathbf{k},\mathbf{q}+\mathbf{k})M_{\lambda_2\mu}(\mathbf{k_1}-\mathbf{q},\mathbf{k})]\\
&\times\{[1-f_\mu(\mathbf{k},T)][1-f_\lambda(\mathbf{k_1},T)]f_{\lambda_1}(\mathbf{q}+\mathbf{k},T)f_{\lambda_2}(\mathbf{k_1}-\mathbf{q},T)\\
&- f_\mu(\mathbf{k},T)f_\lambda(\mathbf{k_1},T)[1-f_{\lambda_1}(\mathbf{q}+\mathbf{k},T)][1-f_{\lambda_2}(\mathbf{k_1}-\mathbf{q},T)]\}.\end{aligned}$$ (A11)

In a next step we will make connection to the golden rule result of the main text. It is straightforward to see that using Eq. (A10) we can rewrite the above expression to yield

$$\begin{aligned}[\partial_T - \nabla_{\mathbf{R}}U(\mathbf{R},T)\nabla_{\mathbf{k}}]f_\mu(\mathbf{k},T) = &\frac{2\pi}{v_F}\int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d^2\mathbf{q}}{(2\pi)^2}\delta(\mu k + \lambda k_1 - \lambda_1|\mathbf{k}+\mathbf{q}| - \lambda_2|\mathbf{k_1}-\mathbf{q}|)\times 4[N|T_{\mu\lambda\lambda_2\lambda_1}(\mathbf{k},\mathbf{k_1},\mathbf{q})|^2\\
&- T_{\mu\lambda\lambda_2\lambda_1}(\mathbf{k},\mathbf{k_1},\mathbf{q})T^*_{\mu\lambda\lambda_1\lambda_2}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})]\{[1-f_\mu(\mathbf{k},T)][1-f_\lambda(\mathbf{k_1},T)]f_{\lambda_1}(\mathbf{q}+\mathbf{k},T)f_{\lambda_2}(\mathbf{k_1}-\mathbf{q},T)\\
&- f_\mu(\mathbf{k},T)f_\lambda(\mathbf{k_1},T)[1-f_{\lambda_1}(\mathbf{q}+\mathbf{k},T)][1-f_{\lambda_2}(\mathbf{k_1}-\mathbf{q},T)]\}.\end{aligned}$$ (A12)

Energy and momentum conservation restricts the valid combinations of particles and holes scattering (see Refs. 11 and 14), which simplifies the above expression. Applying all these simplifications and shifting the variables appropriately we obtain

$$\begin{aligned}[\partial_T - \nabla_{\mathbf{R}}U(\mathbf{R},T)\nabla_{\mathbf{k}}]f_\mu(\mathbf{k},T) = &-\frac{(2\pi)}{v_F}\int \frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\{\delta(k - k_1 - |\mathbf{k}+\mathbf{q}| + |\mathbf{k_1}-\mathbf{q}|)\bar{R}_1\{f_\mu(\mathbf{k},t)f_{-\mu}(\mathbf{k_1},t)[1-f_\mu(\mathbf{k}+\mathbf{q},t)]\\
&\times[1-f_{-\mu}(\mathbf{k_1}-\mathbf{q},t)] - [1-f_\mu(\mathbf{k},t)][1-f_{-\mu}(\mathbf{k_1},t)]f_\mu(\mathbf{k}+\mathbf{q},t)f_{-\mu}(\mathbf{k_1}-\mathbf{q},t)\}\\
&\times\delta(k + k_1 - |\mathbf{k}+\mathbf{q}| - |\mathbf{k_1}-\mathbf{q}|)\bar{R}_2\{f_\mu(\mathbf{k},t)f_\mu(\mathbf{k_1},t)[1-f_\mu(\mathbf{k}+\mathbf{q},t)][1-f_\mu(\mathbf{k_1}-\mathbf{q},t)]\\
&- [1-f_\mu(\mathbf{k},t)][1-f_\mu(\mathbf{k_1},t)]f_\mu(\mathbf{k}+\mathbf{q},t)f_\mu(\mathbf{k_1}-\mathbf{q},t)\}\},\end{aligned}$$ (A13)

where

$$\begin{aligned}\bar{R}_1 = &4N[|T_{+--+}(\mathbf{k},\mathbf{k_1},\mathbf{q})|^2 + |T_{+-+-}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})|^2] - 4T_{+--+}(\mathbf{k},\mathbf{k_1},\mathbf{q})T^*_{+-+-}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})\\
&- 4T^*_{+-+-}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})T_{+--+}(\mathbf{k},\mathbf{k_1},\mathbf{q})\end{aligned}$$ (A14)

and

$$\bar{R}_2 = 4N|T_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{q})|^2 - 4T_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{q})T^*_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q}).$$ (A15)

Performing the appropriate shifts allows to write

$$\bar{R}_1 = 4(N-1)|T_{+--+}(\mathbf{k},\mathbf{k_1},\mathbf{q})|^2 + 4(N-1)|T_{+-+-}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})|^2 + 4|T_{+--+}(\mathbf{k},\mathbf{k_1},\mathbf{q}) - T_{+-+-}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})|^2$$ (A16)

and

$$\bar{R}_2 = 4(N-1)|T_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{q})|^2 + 2|T_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{q}) - T_{++++}(\mathbf{k},\mathbf{k_1},\mathbf{k_1}-\mathbf{k}-\mathbf{q})|^2,$$ (A17)

which establishes the equivalence of Fermi's golden rule and the Keldysh treatment [see Eq. (3.26)].

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